ELSEVIER
Discrete Applied Mathematics 49 (1994) 325–330
DISCRETE APPLIED MATHEMATICS

Total matchings and total coverings of threshold graphs

Uri Peled*, Feng Sun

Department of Mathematics, Statistics, and Computer Science, M/C 249, University of Illinois at Chicago, 851 South Morgan Street, Chicago, IL 60607-7045, USA

Received 18 December 1990; revised 15 January 1991

Abstract

A graph is said to be *threshold* if there exist real numbers a_i associated with its vertices i and a real number b such that $\sum_{i \in S} a_i \leqslant b$ holds if and only if S is a stable set of vertices. A vertex of a graph is said to *cover* itself, its neighboring vertices, and the incident edges. An edge is said to *cover* itself, its neighboring edges, and its endvertices. Alavi, Behzed, Lesniak-Foster and Nordhaus defined a *total matching* as a maximal set of vertices and edges that do not cover each other. A *total covering* is defined as a minimal set of vertices and edges that cover every vertex and edge. We determine the largest size of a total matching and the smallest size of a total covering in a threshold graph. Various related parameters are also found.

1. Introduction

A graph $G = (V, E)$ consists of a nonempty set $V = V(G)$ of vertices and a set $E = E(G)$ of edges, which are two-element subsets of V. See [3] for graph theory terminology not defined here. For $W \subseteq V$, the *induced subgraph* $G[W]$ is the graph (W, F), where F is the set of edges of G with both ends in W. The set of neighbouring vertices of a vertex v is denoted $N(v)$. If $N(v) = V - \{v\}$, then v is a *dominating vertex*.

A *clique* is a subset $Q \subseteq V$ such that $G[Q]$ is a complete graph. The *clique number* $\omega(G)$ is the maximum size of a clique in G. An *independent* or *stable* set P of V is a subset of V such that $G[P]$ is a graph without edges. The *stability number* $\alpha(G)$ is the maximum size of a stable set in G.

All the members of $V(G) \cup E(G)$ are called *elements* of G. A vertex v of G is said to *cover* itself, all edges incident to v, and all vertices adjacent to v. Similarly an edge e of G *covers* itself, the two endvertices of e, and all edges adjacent to e. A set $A_t \subseteq V \cup E$ is called a *total cover* of G if every element of G is covered by some element of A_t, and A_t

*Corresponding author. E-mail: u32799@uicvm.uic.edu.

SSDI 0166-218X(92)00142-J

is minimal. Following the notation of [1], we call the number

$$\alpha_2(G) = \min\{|A_t|\colon A_t \text{ is a total cover of } G\}$$

the *total covering number* of G, and the sets achieving the minimum are called *minimum total covers.*

Two elements in G are called *independent* if neither one covers the other. A set $M_t \subseteq V \cup E$ is called a *total matching* of G if the elements of M_t are pairwise independent, and M_t is maximal. The number

$$\beta_2(G) = \max\{|M_t|\colon M_t \text{ is a total matching of } G\}$$

is called the *total matching number* of G, and the sets achieving the maximum are called *maximum total matchings.*

In addition, two other related numbers are defined:

$$\alpha_2'(G) = \max\{|A_t|\colon A_t \text{ is a total cover of } G\},$$

$$\beta_2'(G) = \min\{|M_t|\colon M_t \text{ is a total matching of } G\}.$$

A graph G is said to be *threshold* if there exists a linear inequality whose solutions in 0, 1 variables are precisely the characteristic vectors of the stable sets of G. Threshold graphs have been studied most thoroughly [2, 4–16]. We determine $\alpha_2(G)$, $\beta_2(G)$, $\alpha_2'(G)$, $\beta_2'(G)$, and related parameters for a threshold graph G.

We use $\lfloor x \rfloor$ and $\lceil x \rceil$ to denote the greatest integer $\leqslant x$ and the least integer $\geqslant x$, respectively.

2. Main results

We start with some useful results about total coverings, complete graphs, and threshold graphs.

Fact 2.1 [1]. *If G is a connected graph of order $n \geqslant 2$, then*

$$1 \leqslant \alpha_2(G) \leqslant \beta_2'(G) \leqslant \beta_2(G) \leqslant \alpha_2'(G) \leqslant n-1,$$

$$\alpha_2(G) \leqslant \lceil n/2 \rceil.$$

Lemma 2.2. *Let G have an induced subgraph H with a dominating vertex. Let A be a set of elements of G that covers H. Then $\alpha_2(H) \leqslant |A|$.*

Proof. Construct a set B as follows. Start with $B = A$. Remove from B the elements that do not cover any element of H. If B has any vertices of $V(G) - V(H)$, remove them from B and add to B a dominating vertex of H if B does not have one already. For each edge $e \in B$, e not in H, do the following: the edge e has exactly one endvertex v in H; remove e from B and add v to B if it is not there already. Now B is contained in H, B covers H, and $|B| \leqslant |A|$. Hence $\alpha_2(H) \leqslant |B| \leqslant |A|$. $\square$

Lemma 2.3. *If G contains a clique K, then $\alpha_2(G) \geqslant \alpha_2(G[K])$.*

Proof. Consider any total covering A_t of $G = (V, E)$. Denote by F the set of edges of A_t joining K to $V - K$, and by L the set of vertices of K covered by F. We have $|L| \leqslant |F|$ because every vertex of L is covered by some edges of F, and every edge of F covers only one vertex of L. Also, F does not cover any elements of $G[K - L]$, and therefore $A_t - F$ covers $G[K - L]$. Then by Lemma 2.2, $|A_t - F| \geqslant \alpha_2(G[K - L])$. Hence

$$|A_t| \geqslant |F| + \alpha_2(G[K - L]) \geqslant |L| + \alpha_2(G[K - L]) \geqslant \alpha_2(G[K]),$$

where the last inequality is true because if C is any total covering of $G[K - L]$, then $L \cup C$ covers $G[K]$. Therefore $\alpha_2(G) \geqslant \alpha_2(G[K])$. □

The following fact is mentioned in [1].

Fact 2.4. $\alpha_2(K_n) = \lceil n/2 \rceil$, *where K_n is the complete graph on n vertices.*

Proof. A maximum matching of K_n, supplemented for odd n by the only uncovered vertex, or an appeal to the more general Fact 2.1, shows that $\alpha_2(K_n) \leqslant \lceil n/2 \rceil$. To show the reverse inequality, consider any total covering $A_t = V_t \cup E_t$ of $K_n = (V, E)$. Since V_t does not cover any edges of $G[V - V_t]$, E_t must span every vertex of $V - V_t$ except for one vertex for odd n. Therefore $|E_t| \geqslant \lfloor |V - V_t|/2 \rfloor$, and consequently

$$|A_t| = |E_t| + |V_t| \geqslant \lfloor (|V| + |V_t|)/2 \rfloor.$$

If $V_t \neq \emptyset$, the right-hand side is at least $\lceil n/2 \rceil$. If $V_t = \emptyset$, then E_t must span every vertex of V without exception, so $|E_t| \geqslant \lceil n/2 \rceil$. □

From Fact 2.4 and Lemma 2.3, we can strengthen the inequality $1 \leqslant \alpha_2(G)$ as follows:

Corollary 2.5. *For any graph G, $\lceil \omega(G)/2 \rceil \leqslant \alpha_2(G)$.*

Fact 2.6 [4]. *A graph $G = (V, E)$ is threshold if and only if there is a partition of V into disjoint sets P, Q, and an ordering $u_1, u_2, \ldots, u_k$ of P, such that no two vertices in P are adjacent, every two vertices in Q are adjacent, and $N(u_1) \supseteq N(u_2) \supseteq \cdots \supseteq N(u_k)$.*

Fact 2.7 [4, 16]. *For any threshold graph G, there exists a partition of V as in Fact 2.6 such that $|Q| = \omega(G)$. This partition can be constructed in time $O(n^2)$, where n is the number of vertices of G.*

Now we are ready to prove the main results of this paper.

Theorem 2.8. *If G is a connected threshold graph, then*

$$\alpha_2(G) = \lceil \omega(G)/2 \rceil.$$

Proof. By Corollary 2.5, we only need to show that $\alpha_2(G) \leqslant \lceil \omega(G)/2 \rceil$. Let P, Q be the partition of V as in Fact 2.7. Since G is a connected threshold graph, Q has a vertex v_1 adjacent to every other vertex in V. We distinguish two cases here.

Case 1: $\omega(G)$ *is even*. The set Q has a vertex v_2 without neighbors in P, for otherwise u_1 could be added to Q. Put $A_t = V_t \cup E_t$, where $V_t = \{v_1\}$ and E_t is a perfect matching of $G[Q - \{v_1, v_2\}]$. It is not hard to see that A_t covers G and no proper subset of A_t does, so A_t is a total cover of G with $|A_t| = \lceil \omega(G)/2 \rceil$.

Case 2: $\omega(G)$ *is odd*. Put $A_t = V_t \cup E_t$, where $V_t = \{v_1\}$ and E_t is a perfect matching of $G[Q - \{v_1\}]$. Then A_t covers G, and no proper subset of A_t does, so A_t is a total cover of G with $|A_t| = \lceil \omega(G)/2 \rceil$.

In both cases we have $|A_t| = \lceil \omega(G)/2 \rceil$, so $\alpha_2(G) \leqslant \lceil \omega(G)/2 \rceil$. □

Since every threshold graph consists of isolated vertices and a connected threshold subgraph, we have the following:

Corollary 2.9. *If G is a threshold graph, then*

$$\alpha_2(G) = \lceil \omega(G)/2 \rceil + \textit{number of isolated vertices in } G.$$

Theorem 2.10. *If $G = (V, E)$ is a threshold graph, then*

$$\beta_2(G) = |V| - \lfloor \omega(G)/2 \rfloor.$$

Proof. First we show that $\beta_2(G) \geqslant |V| - \lfloor \omega(G)/2 \rfloor$. Partition V into P, Q as in Fact 2.7, and construct a total matching of size $|V| - \lfloor \omega(G)/2 \rfloor$ as follows:

Case 1: $\omega(G)$ *is odd*. As before, there is a vertex $v_2 \in Q$ without neighbors in P. Put $M_t = V_t \cup E_t$, where $V_t = P \cup \{v_2\}$ and E_t is a perfect matching of $G[Q - \{v_2\}]$. Then M_t is a total matching of size $|P| + (|Q| + 1)/2 = |V| - \lfloor \omega(G)/2 \rfloor$.

Case 2: $\omega(G)$ *is even*. Put $M_t = V_t \cup E_t$, where $V_t = P$ and E_t is a perfect matching of $G[Q]$. Then M_t is a total matching of size $|P| + |Q|/2 = |V| - \lfloor \omega(G)/2 \rfloor$.

We now show that $\beta_2(G) \leqslant |V| - \lfloor \omega(G)/2 \rfloor$. Consider any total matching M_t of G. Let P, Q be the partition of V as in Fact 2.7, and let E_{PQ} be the set of edges of G between P and Q. Then $|M_t \cap (P \cup E_{PQ})| \leqslant |P|$, for otherwise some elements of M_t cover each other. Also M_t has at most $\lceil |Q|/2 \rceil$ elements of the clique $G[Q]$. Therefore $|M_t| \leqslant |P| + \lceil |Q|/2 \rceil = |V| - \lfloor \omega(G)/2 \rfloor$. □

Theorem 2.11. *If G is a threshold graph of order n with at least one edge, then*

$$\alpha'_2(G) = n - 1.$$

Proof. By Fact 2.1 applied to G without its isolated vertices, $\alpha'_2(G) \leqslant n - 1$, so we only need to show $\alpha'_2(G) \geqslant n - 1$. G has a vertex v_1 adjacent to every other nonisolated vertex of G. Let $A_t = V(G) - \{v_1\}$. Obviously A_t covers G, but no proper subset of A_t covers G, so A_t is a total cover with $|A_t| = n - 1$, and $\alpha'_2(G) \geqslant n - 1$. □

Theorem 2.12. *If G is connected threshold graph of order $n \geqslant 2$, then*

$$\beta'_2(G) = \lceil \omega(G)/2 \rceil.$$

Proof. By Fact 2.1 and Theorem 2.8, we have $\beta_2'(G) \geqslant \alpha_2(G) = \lceil \omega(G)/2 \rceil$. Recall the proof of Theorem 2.8. It is not hard to see that the total covering constructed there is a total matching of size $\lceil \omega(G)/2 \rceil$ in G, so $\beta_2'(G) = \lceil \omega(G)/2 \rceil$. □

Corollary 2.13. *If G is a threshold graph, then*

$$\beta_2'(G) = \lceil \omega(G)/2 \rceil + \textit{number of isolated vertices of } G.$$

3. Related results

By restricting the total coverings and matchings of $G = (V, E)$ to subsets of V or E, we obtain the vertex and edge covering numbers, independence number, and maximum matching. We give below the values of these parameters. The proofs are simple and are omitted.

Let

$$\alpha_0(G) = \min\{|A_t|: A_t \text{ is a total cover of } G \text{ such that } A_t \subseteq V(G)\},$$

$$\beta_0(G) = \max\{|M_t|: M_t \text{ is a total matching of } G \text{ such that } M_t \subseteq V(G)\}.$$

Fact 3.1. *If $G = (V, E)$ is a threshold graph, then*

$$\alpha_0(G) = \omega(G) - 1 + \textit{number of isolated vertices of } G,$$

$$\beta_0(G) = |V| - \omega(G) + 1.$$

Let

$$\alpha_0'(G) = \max\{|A_t|: A_t \text{ is a total cover of } G \text{ such that } A_t \subseteq V(G)\},$$

$$\beta_0'(G) = \min\{|M_t|: M_t \text{ is a total matching of } G \text{ such that } M_t \subseteq V(G)\}.$$

Fact 3.2. *If G is a connected threshold graph of order $n \geqslant 2$, then*

$$\alpha_0'(G) = n - 1,$$

$$\beta_0'(G) = 1.$$

Let

$$\alpha_1(G) = \min\{|A_t|: A_t \text{ is a total cover of } G \text{ such that } A_t \subseteq E(G)\},$$

$$\beta_1(G) = \max\{|M_t|: M_t \text{ is a total matching of } G \text{ such that } M_t \subseteq E(G)\}.$$

Fact 3.3. *If G is a threshold graph, then the size of a maximum matching of G is $\beta_1(G) = \lfloor (\omega(G) + l)/2 \rfloor$, where l is number of edges in a maximum matching of the bipartite graph obtained by dropping the edges of a maximum clique of G. Consequently, $\alpha_1(G) = |V(G)| - \lfloor (\omega(G) + l)/2 \rfloor$.*

Corollary 3.4. *If G is a threshold graph with partition $V = P \cup Q$ as in Fact 2.7, then G has a perfect matching if and only if $|N(u_i)| \geqslant |P| - i + 1$ for $1 \leqslant i \leqslant |P|$, and* $\omega(G) \equiv |P| \pmod 2$.

Let

$$\beta_1'(G) = \min\{|M_t|\colon M_t \text{ is a total matching of } G \text{ such that } M_t \subseteq E(G)\},$$

$$\alpha_1'(G) = \max\{|A_t|\colon A_t \text{ is a total cover of } G \text{ such that } A_t \subseteq E(G)\}.$$

Corollary 3.5. *If $G = (V, E)$ is a connected threshold graph, then*

$$\beta_1'(G) = \lfloor \omega(G)/2 \rfloor,$$

$$\alpha_1'(G) = |V| - \lfloor \omega(G)/2 \rfloor.$$

References

[1] Y. Alavi, M. Behzad, L.M. Lesniak-Foster and E.A. Nordhaus, Total matchings and total coverings of graphs, J. Graph Theory 1 (1977) 135–140.
[2] J.S. Beissinger and U.N. Peled, Enumeration of labelled threshold graphs and a theorem of Frobenius involving Eulerian polynomials, Graphs Combin. 3 (1987) 213–219.
[3] J.A. Bondy and U.S. Murty, Graph Theory with Applications (North-Holland, Amsterdam, 1976).
[4] V. Chvátal and P.L. Hammer, Aggregation of inequalities in integer programming, in: Annals of Discrete Mathematics 1 (North-Holland, Amsterdam, 1977) 145–162.
[5] O. Cogis, Ferrers digraphs and threshold graphs, Discrete Math. 38 (1982) 33–46.
[6] M.C. Golumbic, Threshold graphs and synchronizing parallel processes, in: Combinatorics, Colloquia Mathematica Societatis János Bolyai 17 (North-Holland, Amsterdam, 1978) 419–428.
[7] M.C. Golumbic, Algorithmic Graph Theory and Perfect Graphs (Academic Press, New York, 1980).
[8] P.L. Hammer, T. Ibaraki and U.N. Peled, Threshold numbers and threshold completions, in: Annals of Discrete Mathematics 11 (North-Holland, Amsterdam, 1981) 125–145.
[9] P.L. Hammer, T. Ibaraki and B. Simeone, Threshold sequences, SIAM J. Algebraic Discrete Methods 2 (1981) 39–49.
[10] F. Harary and U.N. Peled, Hamiltonian threshold graphs, Discrete Appl. Math. 16 (1987) 11–15.
[11] P.B. Henderson and Y. Zalcstein, A graph-theoretic characterization of the PV chunk class of synchronizing primitives, SIAM J. Comput. 6 (1977) 88–108.
[12] M. Koren, Extreme degree sequences of simple graphs, J. Combin. Theory Ser. B 15 (1973) 213–224.
[13] E.T. Ordman, Threshold coverings and resource allocation, in: Proceedings 16th Southeastern Conference on Combinatorics, Graph Theory and Computing (1985) 99–113.
[14] J. Orlin, The minimal integral separator of a threshold graph, in: Annals of Discrete Mathematics 1 (North-Holland, Amsterdam, 1977) 415–419.
[15] U.N. Peled, Threshold graph enumeration and series-product identities, in: Proceedings 11th Southeastern Conference on Combinatorics, Graph Theory and Computing (1980) 735–738.
[16] U.N. Peled and M.K. Srinivasan, The polytope of degree sequences, Linear Algebra Appl. 114/115 (1989) 349–377.